2-mar Lecture 9/Review

Monday, March 2, 2026 9:24 AM

explore example w constant coefficients:

$ay'' + by' + cy = 0$ where a, b, c are constant coeff.

try $y = e^{rx}$ then $y' = re^{rx}$

$y'' = r^2 e^{rx}$

ex $y'' - 5y' + 6y = 0$ becomes

$r^2 e^{rx} - 5re^{rx} + 6e^{rx} = 0$

$e^{rx}(r^2 - 5r + 6) = 0$

e^{rx}: never zero

$(r-2)(r-3) = 0$

$r = 2$ or $r = 3$ $\xrightarrow{\text{2 sol'ns}}$ $y_1(x) = e^{2x}$, $y_2(x) = e^{3x}$

back to general example:

$ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0$

$e^{rx}(ar^2 + br + c) = 0$

conclude: $y = e^{rx}$ will satisfy DE $ay'' + by' + cy = 0$ precisely when r is a root of $\underbrace{ar^2 + br + c = 0}$

called characteristic equation of $ay'' + by' + cy = 0$

if $ar^2 + br + c = 0$ has 2 distinct roots r_1, r_2 then corresponding sol'ns $y_1(x) = e^{r_1 x}$ and $y_2(x) = e^{r_2 x}$ are linearly independent

Thm: if real, distinct roots of characteristic eq'n are r_1, r_2 then $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ is general sol'n of $ay'' + by' + cy = 0$

ex find general sol'n of $2y'' - 7y' + 3y = 0$

$$ay'' + by' + cy = 0$$

ex. find general sol'n of $2y'' - 7y' + 3y = 0$

corresponding char. eqn: $2r^2 - 7r + 3 = 0$

$$(2r-1)(r-3) = 0$$

$$r_1 = \tfrac{1}{2} \quad r_2 = 3$$

then general sol'n is $\boxed{y(x) = c_1 e^{\frac{1}{2}x} + c_2 e^{3x}}$

ex. find gen. sol'n of $y'' + 2y' = 0$

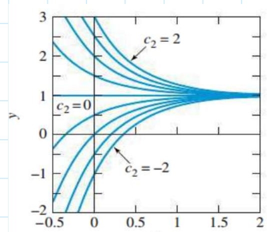
$$r^2 + 2r = 0$$

$$r(r+2) = 0$$

$$r_1 = 0 \qquad r_2 = -2$$

$$e^0 = 1 \qquad e^{-2x} \implies \boxed{y(x) = c_1 + c_2 e^{-2x}}$$

when $c_1 = 1$ solution curves approach $y(x) \equiv 1$ as $x \to +\infty$



Repeated Roots

if $r_1 = r_2$ then have single sol'n $y_1(x) = e^{r_1 x}$

and 2^{nd} sol'n would be $y_2(x) = xe^{r_1 x}$

ex. solve IVP $y'' + 2y' + y = 0$ given $y(0) = 5$, $y'(0) = -3$

$$r^2 + 2r + 1 = 0$$

$$(r+1)^2 = 0 \implies r_1 = r_2 = -1$$

gen. soln: $y(x) = c_1 e^{-x} + c_2 x e^{-x}$

$$y(0) = c_1 + 0 = 5 \implies c_1 = 5$$

$$y(x) = 5e^{-x} + c_2 x e^{-x}$$

$$y'(x) = -5e^{-x} + c_2(e^{-x} \mp xe^{-x})$$

$y'(x) = -5e^{-x} + c_2\left(e^{-x} \mp xe^{-x}\right)$

$y'(0) = -5 + c_2(1 - 0) = -3$

$c_2 = 2$

$\therefore \boxed{y(x) = 5e^{-x} + 2xe^{-x}}$

the above material from Section 3.1 will <u>NOT</u> be on Exam 1

types of equations seen so far:

- separable
- ones use an integrating factor w 1ST order <u>linear</u> DE

$$\frac{dy}{dx} + \underbrace{P(x)y}_{\downarrow} = Q(x)$$

$$\rho = e^{\int P(x)\,dx}$$

"manufacture" Product Rule on LHS

$(fg)' = f'g + fg'$

$D_x[fg]$

- homogeneous 1ST order DE $\quad \frac{dy}{dx} = F\left(\frac{y}{x}\right)$

substitution $\Rightarrow v = \frac{y}{x} \Rightarrow y = \overset{f}{v}\overset{g}{x}$

$$\frac{dy}{dx} = \overset{f'g}{\frac{dv}{dx}x} + \overset{fg'}{v}_{(1)}$$

- Bernoulli Eq'n $\quad \frac{dy}{dx} + P(x)y = Q(x)y^n \searrow v = y^{1-n} \quad$ if $n=0$ or $n=1$

use integrating factor

- reducible $\quad$ format $F(x, y, y', y'') = 0 \searrow$ becomes

$F(x, y, y', y'') = 0$ becomes $F(x, p, p') = 0$

y missing: $p = y' = \frac{dy}{dx}$ $\quad y'' = \frac{dp}{dx}$

x missing: $p = y' = \frac{dy}{dx}$ $\quad y'' = \frac{dp}{dy} \cdot \frac{dy}{dx}$

$$= \frac{dp}{dy} \cdot p$$

$F(x, y, y', y'') = 0$ becomes $F\left(y, p, p\frac{dp}{dy}\right) = 0$

- autonomous eqn $\frac{dx}{dt} = f(x)$ no y

  critical pts (CPs) are important $f(x) = 0$

  $\hookrightarrow$ equilibrium sol'ns

- homogeneous 2^{nd} order DE (linear)

  format: $A(x)y'' + B(x)y' + C(x)y = F(x)$

  $A(x) \neq 0$

  (divide): $y'' + p(x)y' + q(x)y = f(x)$

  associated homogeneous eqn: $y'' + p(x)y' + q(x)y = 0$

  if $y_1(x)$, $y_2(x)$ are solutions then so is $y(x) = c_1 y_1 + c_2 y_2$

use a Wronskian to determine if y_1, y_2 are linearly independent or linearly dependent

$$W = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} \neq 0 \Rightarrow f, g \text{ are linearly independent}$$

$$= 0 \Rightarrow f, g \text{ are linearly dependent}$$

$= 0 \Rightarrow f, g$ are linearly dependent

Double Ange Identities

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$= 1 - 2\sin^2\theta$

$= \cos^2\theta - \sin^2\theta$